Emergence of stability in a stochastically driven pendulum: Beyond the Kapitsa pendulum

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We consider a prototypical nonlinear system which can be stabilized by multiplicative noise: an underdamped nonlinear pendulum with a stochastically vibrating pivot. A numerical solution of the pertinent Fokker-Planck equation shows that the upper equilibrium point of the pendulum can become stable even when the noise is white, and the "Kapitsa pendulum" effect is not at work. The stabilization occurs in a strong-noise regime where WKB approximation does not hold.

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It has been known for a long time that multiplicative noise can enhance stability of nonlinear systems. Examples are numerous indeed and culminate at noise-induced phase transitions far from equilibrium [1]. This Brief Report deals with a noise-induced stabilization of oscillating systems. As a prototypical example we consider an underdampled nonlinear pendulum with a stochastically vibrating pivot. The stochastic driving introduces both multiplicative and additive noise; see Fig. 1. Our numerical simulations clearly show that the multiplicative noise can stabilize the otherwise unstable upper equilibrium point of the pendulum. The mechanism for this stabilization is markedly different from, and more subtle than, the "Kapitsa pendulum" mechanism. The Kapitsa pendulum involves a (deterministic) monochromatic parametric driving of the pendulum at a frequency that is much higher than the natural frequency of the pendulum [2]. Here the upper equilibrium point becomes stable if the driving acceleration is higher than a critical value depending on the pendulum length and the gravity acceleration. In the Kapitsa pendulum problem the change in stability of the upper equilibrium point comes from a change in the effective potential of the pendulum [2].

Extensions of the Kapitsa pendulum effect to multiplicative stochastic driving have also been considered [3,4]; see Ref. [5]. for a review. In these extensions the noise spectrum is strongly peaked at a single frequency which is much higher than the natural frequency of the pendulum. The presence of a *high-frequency* noise of a sufficient strength modifies the effective potential, and this can stabilize the upper equilibrium point. In the theoretical treatment, this setting introduces a time-scale separation which permits a perturbative approach [5]. In this work we consider a model stochastic driving with a flat spectrum: a white noise. Here all frequencies from 0 to ∞ are equally present, and there is no time-scale separation. We will show that, for such a noise, the upper position of the pendulum can also become stable. However, the stabilization cannot be traced to a change in the effective potential of the pendulum.

A stochastically driven simple gravity pendulum can be described by a Langevin equation,

$$\dot{\theta} = \Omega, \tag{1}$$

$$\dot{\Omega} = -\omega_0^2 \sin\theta - 2\gamma\Omega + \frac{1}{l}\sin\theta\sqrt{\mu}\xi_1(t) + \sqrt{\alpha}\xi_2(t), \tag{2}$$

where θ is the deviation angle of the pendulum; see Fig. 1, Ω is the angular velocity, $\omega_0 = \sqrt{g/l}$ is the harmonic frequency of the pendulum, γ is the damping factor, l is the pendulum length, g is the gravity acceleration, and μ and α are the magnitudes of the multiplicative and additive noises ξ_1 and ξ_2, respectively. The noises are assumed to be Gaussian, white with zero mean, and mutually uncorrelated,

$$\langle \xi_i(t)\rangle = 0, \quad \langle \xi_i(t)\xi_j(t')\rangle = 2\delta(t-t')\delta_{i,j}, \quad i,j = 1,2. \tag{3}$$

Langevin Eqs. (1) and (2) are equivalent (see, e.g., Ref. [6]) to the following Fokker-Planck equation for the probability distribution $W(\theta,\Omega,t)$:

$$W_t = -\Omega W_\theta + \omega_0^2 \sin\theta W_\Omega + 2\gamma\frac{\partial}{\partial\Omega}(\Omega W) + \left(\alpha + \frac{\mu}{l^2}\sin^2\theta\right)W_{\Omega\Omega}, \tag{4}$$

where the indices θ, Ω and t denote the corresponding partial derivatives of $W(\theta,\Omega,t)$. As the noises are Ω independent, there is no difference between the Ito and Stratonovich interpretations. Introducing the dimensionless variables $\tilde{t} = \omega_0 t$, $\tilde{\Omega} = \Omega/\omega_0$, and $\tilde{W}(t,\theta,\Omega) = \omega_0 W(t,\theta,\Omega)$, we can rewrite the Fokker-Planck equation in a dimensionless form,

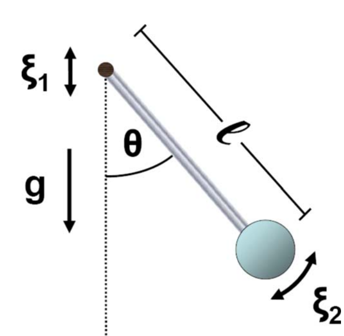

FIG. 1. (Color online) Schematic of the stochastically driven simple gravity pendulum.

$$W_t = -\Omega W_\theta + \sin\theta W_\Omega + 2\Gamma\frac{\partial}{\partial\Omega}(\Omega W) + (\varepsilon + \delta\sin^2\theta)W_{\Omega\Omega}, \tag{5}$$

where $\Gamma=\gamma/\omega_0$, $\varepsilon=\alpha/\omega_0^3$, and $\delta=\mu/(l^2\omega_0^3)$ are the rescaled parameters of the system, and the tildes are omitted.

We assume that, after a transient, the stochastic system approaches a smooth steady state for which $W(\theta,\Omega,t)$ is independent of time: $W(\theta,\Omega,t\to\infty)=\bar{W}(\theta,\Omega)$. The steady-state probability distribution $\bar{W}(\theta,\Omega)$ obeys the equation

$$-\Omega\bar{W}_\theta + \sin\theta\bar{W}_\Omega + 2\Gamma\frac{\partial}{\partial\Omega}(\Omega\bar{W}) + (\varepsilon + \delta\sin^2\theta)\bar{W}_{\Omega\Omega} = 0. \tag{6}$$

We classify a point (θ,Ω) as a stable point of the system if it is a local maximum of the stationary probability distribution $\bar{W}(\theta,\Omega)$. The necessary and sufficient conditions for a function of two variables $f(x,y)$ to have a local maximum at (x_0,y_0) are (see, e.g., Ref. [7]),

$$f_x(x_0,y_0) = f_y(x_0,y_0) = 0, \tag{7}$$

$$f_{xx}(x_0,y_0) < 0 \ \text{ or } \ f_{yy}(x_0,y_0) < 0, \tag{8}$$

$$f_{xx}(x_0,y_0)f_{yy}(x_0,y_0) - f_{xy}^2(x_0,y_0) > 0, \tag{9}$$

where the indices x and y denote partial derivatives. Let us examine the stability properties of the upper equilibrium point ($\theta=\pi$, $\Omega=0$) of the driven pendulum. Equation (6) is invariant under the transformation $\theta\to 2\pi-\theta$, $\Omega\to-\Omega$, that is under a reflection of the axes θ and Ω around the point $(\pi,0)$. Its solution $\bar{W}(\theta,\Omega)$ must obey the same symmetry. Therefore, the first derivatives $\bar{W}_\theta$ and $\bar{W}_\Omega$ must vanish at $(\pi,0)$, and so Eqs. (7) are satisfied there. This immediately follows

$$\bar{W}_{\Omega\Omega}(\pi,0) = -\frac{2\Gamma}{\varepsilon}\bar{W}(\pi,0) < 0,$$

so Eq. (8) is also satisfied at $(\pi,0)$. As a result, the necessary and sufficient condition for $(\pi,0)$ to be a stable point is given by Eq. (9),

$$\Delta \equiv \bar{W}_{\Omega\Omega}(\pi,0)\bar{W}_{\theta\theta}(\pi,0) - \bar{W}_{\theta\Omega}^2(\pi,0) > 0. \tag{10}$$

For $\delta=0$ (only additive noise), steady-state Eq. (6) is soluble analytically (see Ref. [8]),

$$\bar{W}(\theta,\Omega) = \frac{\Gamma^{1/2}}{2\pi^{3/2}\varepsilon^{1/2}I_0(2\Gamma/\varepsilon)}\exp\left[-\frac{\Gamma}{\varepsilon}(\Omega^2 - 2\cos\theta)\right], \tag{11}$$

where $I_0(\dots)$ is the modified Bessel function. In this case the point $(\pi,0)$ is unstable, as the stability parameter Δ, defined in Eq. (10), is negative,

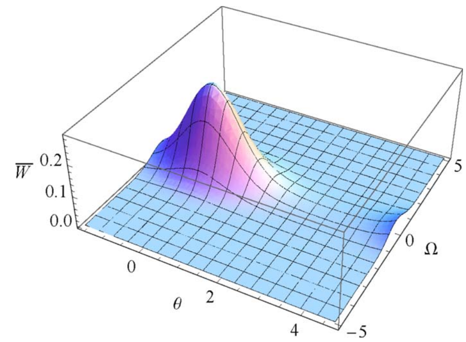

FIG. 2. (Color online) A steady-state probability distribution with no multiplicative noise. The parameters are $\varepsilon=0.1$, $\Gamma=0.1$, and $\delta=0$. This solution evolved from the initial condition $W=(2\pi\sqrt{\pi})^{-1}\exp(-\Omega^2)$ after 60 normalized time units. It coincides with analytical solution (11).

$$\Delta_0 = -\frac{\Gamma^3}{\pi^3\varepsilon^3 I_0^2(2\Gamma/\varepsilon)}\exp\left(-\frac{4\Gamma}{\varepsilon}\right) < 0. \tag{12}$$

Probability distribution (11) is depicted in Fig. 2.

What happens when $\delta>0$? First, we show that there is no stabilization in the weak-noise limit, $\delta\sim\varepsilon\ll 1$, Γ. In this limit one can use a dissipative variant of WKB approximation (see e.g., Ref. [6]) and make the ansatz

$$\bar{W}(\theta,\Omega) = A(\theta,\Omega)\exp\left[-\frac{S(\theta,\Omega)}{\varepsilon}\right],$$

assuming that the prefactor A varies on scales much larger than $1/\varepsilon$. In the leading order in $1/\varepsilon$ one obtains

$$\Omega S_\theta - \sin\theta S_\Omega - 2\Gamma\Omega S_\Omega + \left(1 + \frac{\delta}{\varepsilon}\sin^2\theta\right)S_\Omega^2 = 0. \tag{13}$$

This first-order partial differential equation for the action $S(\theta,\Omega)$ has the form of a stationary Hamilton-Jacobi equation (see e.g., Ref. [9]) with the Hamiltonian $H(\theta,\Omega,p_1,p_2)$ of the form

$$H = \Omega p_1 - \sin\theta p_2 - 2\Gamma\Omega p_2 + \left(1 + \frac{\delta}{\varepsilon}\sin^2\theta\right)p_2^2. \tag{14}$$

Here $p_1=S_\theta$ and $p_2=S_\Omega$ are the canonical momenta conjugated to the coordinates θ and Ω, respectively. As Hamilton-Jacobi Eq. (13) is stationary, we are interested in the zero-energy dynamics $H=0$. The Hamilton's equations

$$\dot{\theta} = \Omega, \quad \dot{\Omega} = -\sin\theta - 2\Gamma\Omega + 2\left(1 + \frac{\delta}{\varepsilon}\sin^2\theta\right)p_2, \tag{15}$$

$$\dot{p}_1 = \cos\theta p_2 - \frac{\delta}{\varepsilon}\sin 2\theta p_2^2, \quad \dot{p}_2 = -p_1 + 2\Gamma p_2, \tag{16}$$

have two zero-energy fixed points: $a_1=(0,0,0,0)$ and $a_2=(\pi,0,0,0)$, corresponding to the lower and upper equilibrium points of the pendulum, respectively. The linear stability of the upper equilibrium point a_2 is determined by the quadratic approximation to the Hamiltonian around a_2,

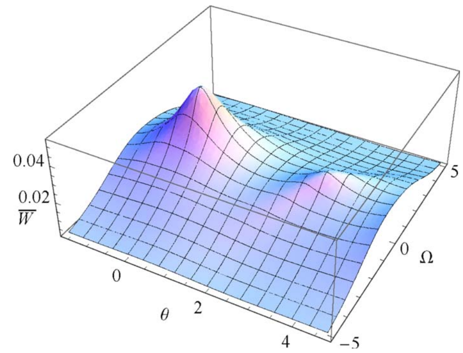

FIG. 3. (Color online) Multiplicative noise stabilizes the upper equilibrium point of the pendulum and makes the pendulum bistable. Shown is the steady-state probability distribution $\bar{W}(\theta,\Omega)$, which exhibits a local maximum at the upper equilibrium point ($\theta=\pi$, $\Omega=0$). The parameters are $\varepsilon=0.1$, $\Gamma=0.1$, and $\delta=2$. The steady-state probability distribution was found by solving numerically time-dependent Fokker-Planck Eq. (5). It evolved from initial condition (11) after 60 dimensionless time units.

$$H(\theta,\Omega,p_1,p_2) \simeq \Omega p_1 + \theta p_2 - 2\Gamma\Omega p_2 + p_2^2. \qquad (17)$$

This quadratic approximation is noiseless, as the noise term in Eq. (14), $(\delta/\varepsilon)\sin^2\theta p_2^2$, is biquadratic in small deviations from the fixed point. Therefore, a weak multiplicative noise does not generate any correction to the potential of the pendulum, in agreement with an early observation [3], and cannot change the stability properties of the system.

However, for a strong multiplicative noise numerical solutions of time-dependent Fokker-Planck Eq. (5) do show emergence of stability of the upper equilibrium point. We obtained these numerical results with a "Mathematica" partial differential equations solver. The numerical domain was $0<\theta<2\pi$ and $-\Omega_{\max}<\Omega<\Omega_{\max}$ with $\Omega_{\max}$ chosen, separately for each set of parameters, sufficiently large. Periodic boundary conditions in θ were used. As to the boundary conditions in Ω, we checked that, at sufficiently large $\Omega_{\max}$, the steady-state solution remained the same up to 1 per cent, whether we imposed periodic or zero-W conditions at the boundary. Larger $\Omega_{\max}$ needed to be taken when Γ became smaller than ε and when δ became large compared to Γ and ε. The values of $\Omega_{\max}$ that we used ranged from $\Omega_{\max}=2$ for $\Gamma=0.5$, $\varepsilon=0.05$, and $\delta\simeq0.15$ to $\Omega_{\max}=15$ for $\Gamma=0.025$, $\varepsilon=0.5$, and $\delta\simeq0.43$. After verifying that $\Omega_{\max}$ is sufficiently large, we used the zero-W boundary conditions in Ω, as this choice reduced the computational time. For each set of parameters δ, Γ and ε we used analytical solution (11) for $\delta=0$ as the initial condition.

We ran the solution until $t=t_{\max}$ such that the value of Δ, evaluated at $t=t_{\max}$, was within 1 per cent from its value at $t=t_{\max}/2$. The larger the parameter Γ, the smaller $t_{\max}$ was needed. The values of $t_{\max}$ that we used ranged from 40 for $\Gamma=1$, $\varepsilon=0.3$, and $\delta=0.6$ to 140 for $\Gamma=0.025$, $\varepsilon=0.05$, and $\delta=0.15$.

At small Γ/ε or large δ the steady-state probability distribution broadens which demands a larger $\Omega_{\max}$, a longer computation time and more computer memory. At large Γ/ε or small δ the distribution becomes too narrow to numerically resolve it with confidence. We verified that, starting from an arbitrary initial condition, the probability function always converges to the same steady-state solution. We also checked that for $\delta=0$ the steady-state solution coincides, with a high accuracy, with analytical solution (11).

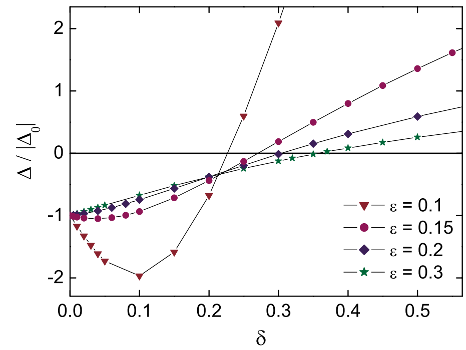

FIG. 4. (Color online) The stability parameter Δ, normalized to $|\Delta_0|$ from Eq. (12), is plotted versus the rescaled magnitude of the multiplicative noise δ for different values of ε at $\Gamma=0.1$. The lines are only given for guiding the eyes. The crossing point of each plot with the line $\Delta=0$ yields δ_c.

Figure 3 gives a typical numerical example of a steady-state probability distribution having a distinct local maximum at $(\pi,0)$, in addition to the expected (higher) peak at (0,0). Also noticeable is a significant broadening of the probability distribution, compared with the case of only additive noise (see Fig. 2).

To determine the stability properties of the upper equilibrium point $(\pi,0)$, we plotted the stability parameter Δ, defined in Eq. (10), versus the rescaled magnitude of the multiplicative noise δ, for different values of Γ and ε. Several examples of such plots are shown in Fig. 4. Using interpolation, we found the critical values $\delta_c=\delta_c(\Gamma,\varepsilon)$ of δ, at which $\Delta=0$.

Figure 5 shows the plot of δ_c versus Γ at different ε. One can see that δ_c slowly decreases with decreasing Γ and approaches a finite value at $\Gamma\to0$. The plot of δ_c versus ε at different Γ, presented in Fig. 6, shows a much stronger, square-root-like dependence at small ε. Our data for different Γ do not contradict a power-law behavior $\delta_c\sim\varepsilon^{\alpha}$ with $\alpha\simeq0.44$, as shown in Fig. 7 for $\Gamma=0.1$. When ε approaches 0, δ_c also goes to zero, and one would expect to always see

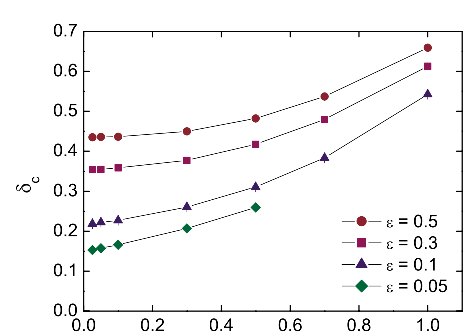

FIG. 5. (Color online) The critical rescaled magnitude of the multiplicative noise δ_c vs the rescaled damping factor Γ for different values of ε. The lines are only given for guiding the eyes.

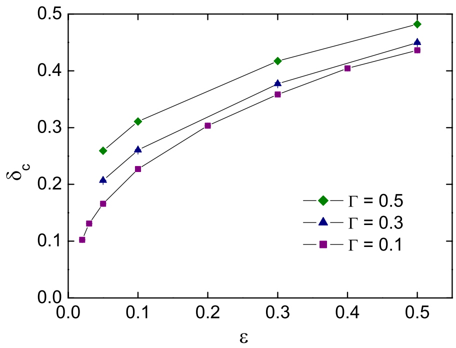

FIG. 6. (Color online) The critical rescaled magnitude of the multiplicative noise δ_c vs the rescaled magnitude of the additive noise ε for different values of Γ. The lines are only given for guiding the eyes.

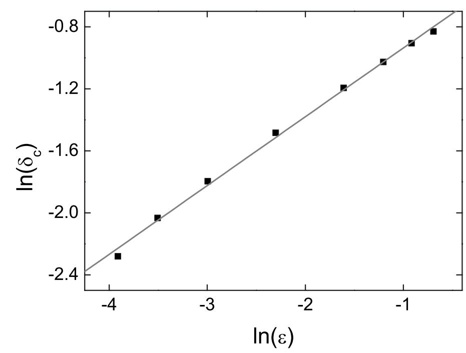

FIG. 7. The log-log plot of δ_c vs ε for $\Gamma=0.1$. A linear fit gives a slope of 0.44 ± 0.01 with the coefficient of determination $R^2=0.997\,26$.

stability in this case. As $\varepsilon\to 0$, however, the probability distribution develops singularities both at (0,0), and at $(\pi,0)$, but the peak at (0,0) becomes much higher than that at $(\pi,0)$. A local maximum at $(\pi,0)$ is physically insignificant in this case. Only with a sufficiently large additive noise the probability distribution becomes sufficiently broad to allow a reasonable probability for the pendulum to be at and around $(\pi,0)$.

In summary, we have investigated the stabilization of a prototypical nonlinear oscillating system by a multiplicative white noise. The stabilization is clearly observed in the numerical solution of the Fokker-Planck equation and requires a supercritical noise magnitude. The stabilization cannot be traced to a change in the effective potential of the system and is not predicted by a WKB analysis which assumes a weak noise. The predicted effect should be observable in experiment. A stochastically driven pendulum will always spend more time around its *lower* equilibrium point. Therefore, to observe the stabilization effect, one should track the deviation angle of the pendulum for a sufficiently long time and analyze the residence time histogram as a function of the deviation angle and angular momentum.

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